Planarity and the Four-Color Theorem

In the previous supplemental notes on an Intro to Graph Theory the definition of planar graphs and Euler's Polyhedron Theorem were covered. Here we discuss Kuratowski's characterization of nonplanar graphs and the Four-Color Theorem

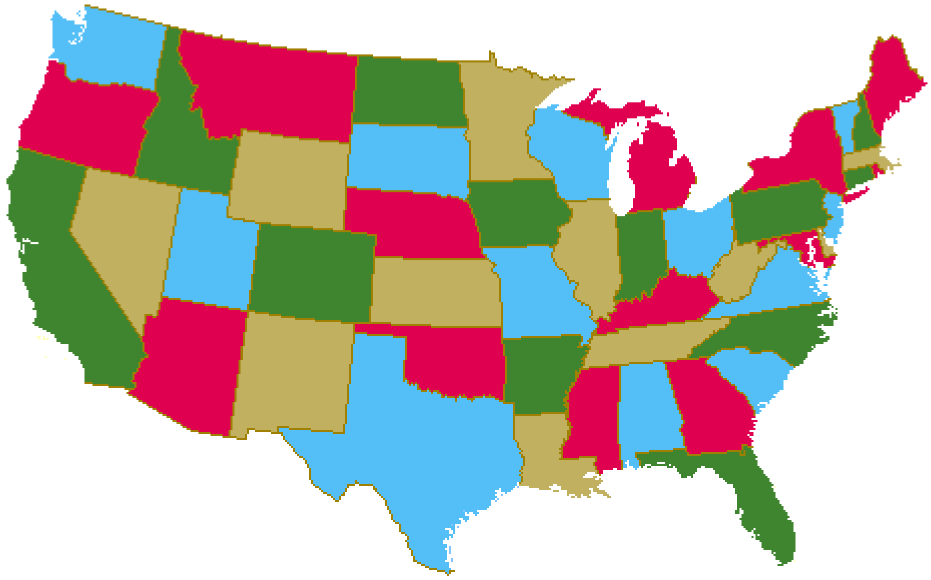
A graph is *bipartite* if there exists a bipartition of the vertex set V into two sets X and Y such that every edge has one end in X and the other in Y. The *complete bipartite* graph $K_{i,j}$ is the bipartite graph with $n = i + j$ vertices, i vertices belonging to X and j vertices belonging to Y, such that there is an edge joining every vertex $x \in X$ to every vertex $y \in Y$.

The graphs K_5 and $K_{3,3}$ are both nonplanar graphs. If you try drawing them in the plane without edge crossings, you will have trouble. In fact, K_5 and $K_{3,3}$ are in some sense the quintessential nonplanar graphs (see Theorem below). A *subdivision* S of G is a graph obtained from G by replacing each edge e with a path joining the same two vertices as e Observe that G is a subdivision of itself (replace each edge with a path of length 1). Clearly, if G is nonplanar, then every subdivision of G is nonplanar. It is also clear that if G contains a nonplanar graph, then G is nonplanar. It follows that a graph is nonplanar if it contains a subgraph that is a subdivision of K_5 or $K_{3,3}$. The fact that the converse is also true is a surprising and deep result discovered by Kuratowski. We state Kuratowski's theorem without proof.

KURATOWSKI'S THEOREM.

A graph G is nonplanar if, and only if, it contains a subgraph that is (isomorphic to) a subdivision of K_5 or $K_{3,3}$.

The condition of being planar occurs in many computer applications, such as the design of VLSI circuits. Moreover, planar graphs became of interest long before the advent of computers. For example, in 1750 Euler gave his famous polyhedron formula relating the number of vertices, edges, and faces of a connected planar graph. As another example, a very famous mathematical question known as the four-color conjecture can be modeled using planar graphs. For centuries map makers had known implicitly that they only needed four colors to color the countries in any map drawn on the globe so that no two neighboring countries (that is, countries sharing a common boundary consisting of more than an isolated point) get the same color (called a *proper* coloring). See Figure below for a four-coloring of the map of the 48 contiguous states of the U.S.A. That four colors suffice for any map on the globe was stated by Guthrie in 1856 as a formal conjecture, which resisted proof for over 100 years.



A proper (face) 4-coloring of the map of the USA (color Alaska and Hawaii as you like)

Given any map on the globe, there is a naturally associated planar graph whose vertices correspond to the countries, and where two vertices are adjacent if, and only if, the countries they represent are neighbors. It is easy to see that the graph associated with a map on the globe is planar. The map coloring problem then transforms via the dual graph to the equivalent problem of coloring the vertices of a planar graph using no more than four colors so that no two adjacent vertices get the same color. Certainly, the transformed problem has a rather elegant mathematical formulation unencumbered by geometrically complex boundary curves associated with maps. Moreover, graphs can be input to a computer using simple data structures such as adjacency matrices, so proofs involving exhaustive case checking are sometimes possible.

In fact, it was just that type of proof (together with deep mathematical insight) that was used by Appel and Haken in 1970 to settle the four-color conjecture in the affirmative. The proof consisted in showing that two contradictory conditions hold for planar graphs that are not four-colorable: there is a certain finite set of planar graphs *S* (at least) one of which would have to occur as a subgraph of any planar graph *G* that is not four-colorable (*S* is an *unavoidable* set), and if a graph *G* contains one of these subgraphs, then there would be another graph *G'* on fewer vertices that is also not four-colorable (each graph in *S* is *reducible*).

Clearly, these two conditions rule out the existence of a planar graph that is not four-colorable, since if such things exist there would have to be one having a minimal number of vertices amongst all such graphs. Appel and Haken proved condition 1 mathematically, but used the computer to check that each graph in *S* had the reducibility property given in condition 2. Since then other proofs of the four-color conjecture have been given along the same lines but using a smaller set of unavoidable reducible graphs. However, exhaustive computer checking of cases remains a component of all known proofs of the four-color conjecture.